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Universality and the renormalisation group

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Abstract: Several functional renormalisation group (RG) equations including Polchinski flows and Exact RG flows are compared from a conceptual point of view and in given truncations. Similarities and differences are highlighted with special emphasis on stability properties. The main observations are worked out at the example of O(N) symmetric scalar field theories where the flows, universal critical exponents and scaling potentials are compared within a derivative expansion. To leading order, it is established that Polchinski flows and ERG flows — despite their inequivalent derivative expansions — have *identical* universal content, if the ERG flow is amended by an adequate optimisation. The results are also evaluated in the light of stability and minimum sensitivity considerations. Extensions to higher order and further implications are emphasized.

Keywords: Nonperturbative Effects, Field Theories in Lower Dimensions, Renormalization Group.

Contents

1. Introduction

It is an experimental fact that physical correlation lengths are diverging in the vicinity of a critical point like a second order phase transition. The absence of dimensionful length scales implies scale invariance of physical correlation functions. Then, properties of physical systems close to a scaling regime or a critical point are characterised by universal exponents and scaling laws [1].

In quantum field theory, functional methods like the renormalisation group are important tools in the study of strongly coupled systems and critical phenomena. The wilsonian renormalisation group is based on the successive integrating-out of momentum degrees of freedom from a path integral representation of the theory, thereby interpolating between a given classical theory and the full quantum effective action [2 – 5] (for reviews, see [6]). The strength of these methods is their flexibility when it comes to approximations, in particular for theories with strong couplings or large correlation lengths. Furthermore, powerful optimisation criteria are available to increase the reliability within given truncations [7 – 9]. Wilsonian flows play an important role in the study of universal scaling phenomena in gauge theories and gravity [10 – 12].

In this Letter, we compare different implementations of a wilsonian cut-off, the Exact Renormalisation Group based on an infrared momentum cut-off for the full effective action Γ, and the Polchinski renormalisation group based on an ultraviolet momentum cut-off for the action S. Both approaches correspond to exact flows, meaning that the endpoint of the fully integrated flow is given by the physical theory. This equivalence, in general, is lost in given truncations due to the qualitative differences in these approaches. We analyse the structural differences both in the full flows and within a derivative expansion. We also establish the remarkable result that the universal information encoded within the Polchinski renormalisation group to leading order in a derivative expansion is equivalent to the Exact Renormalisation Group, if the latter is amended by an adequate optimisation. Extensions to higher order and further implications of this result are equally discussed.

2. Generalities

Wilsonian flows are based on integrating-out momentum modes from a path integral representation of quantum field theory. A wilsonian flow connects a short-distance effective action - typically the classical action - with the full physical theory. Given that the main physical information is contained in the integrated flow, the key properties of the different implementations proposed in the literature deserve a more detailed study.

The Exact Renormalisation Group is based on a cutoff term $\Delta S_k = \frac{1}{2}\int \phi\, R\, \phi$, added to the Schwinger functional. The operator $R(q)$ introduces a momentum cutoff at momentum scale $q^2 \approx k^2$. This induces a k-dependence on the level of the effective action. In its modern form, the flow for an effective action Γ_k for bosonic fields ϕ is given by the simple one-loop expression [3 – 5]

$$\partial_t \Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\,\frac{1}{\Gamma_k^{(2)} + R}\,\partial_t R\,. \tag{2.1}$$

Here, $t \equiv \ln k$ is the logarithmic scale parameter, and the trace denotes a loop integration and a sum over fields and indices, and $\Gamma_k^{(2)}[\phi](p,q) \equiv \delta^2\Gamma_k/\delta\phi(p)\delta\phi(q)$. The cutoff function R obeys $R(q^2) \to 0$ as $k^2/q^2 \to 0$, $R(q^2) > 0$ as $q^2/k^2 \to 0$, and $R(q^2) \to \infty$ as $k \to \Lambda$, and can be chosen freely elsewise (see [7, 13] for more details). This ensures that the flow is well-defined, thereby interpolating between an initial action at $k = \Lambda$ in the ultraviolet (UV) and the full quantum effective action in the infrared (IR). In momentum space, the flow (2.1) receives its main contributions for momenta in the vicinity of $q^2 \approx k^2$, because large momentum modes are suppressed by $\partial_t R$, and small momentum modes are suppressed because R is an IR cutoff.

The flow (2.1) depends on fields and couplings only through the full propagator. More generally, any exact RG flow for Γ_k with a one-loop structure always depends linearly on the full propagator, and hence on the inverse of $\Gamma_k^{(2)}$ [13]. The linear dependence on the full propagator implies that an exact flow for Γ_k is small whenever the full propagator is quantitatively small in the momentum regime where $\partial_t R$ is peaked, e.g. for large fields or strong coupling. Hence, the wilsonian flow (2.1) is essentially local both in field- and momentum-space. Owing to this structure, the flow is amiable to truncations local in the fields (vertex functions) in the range where (2.1) is non-trivial. An integration of the flow in given truncations requires a good control over the full propagator. The stability of (2.1) — in the regions of momentum- and field-space where it is non-trivial — is controlled by the cutoff propagator, and hence by the momentum cutoff R [7]. Therefore, the convergence of truncated flows can be increased by appropriate choices of R. Efficient optimisation criteria based on the flow are available [7 − 9].

A different version of an exact renormalisation group has been introduced by Polchinski [2], and is based on an ultraviolet regulator $K(q^2/\Lambda^2)$ for propagators in the path integral, where Λ denotes the ultraviolet scale parameter. The Polchinski flow equation for the wilsonian action $S_\Lambda[\phi]$ for a scalar field theory is given by [14]

$$\partial_t S_\Lambda[\phi] = \frac{1}{2}\,\mathrm{Tr}\;\partial_t P_\Lambda(q)\,\left[S^{(1)}_\Lambda(q)\,S^{(1)}_\Lambda(-q) - S^{(2)}_\Lambda(q,-q) - 2(P_\Lambda(q))^{-1}\phi(q)\,S^{(1)}_\Lambda(q)\right], \quad (2.2)$$

where $S^{(1)}[\phi](q) \equiv \delta S[\phi]/\delta\phi(q)$, $P_\Lambda(q) = K(q^2/\Lambda^2)/q^2$ and $t \equiv \ln\Lambda$. The cutoff function K is chosen such that high momentum modes are suppressed, $K(q^2/\Lambda^2) \to \infty$ as $q^2/\Lambda^2 \to 0$. The scale dependence introduced via K induces the scale dependence of $S_\Lambda[\phi]$ on Λ. By construction, the flow equation (2.2) is exact in the same sense as the ERG flow (2.1).

The Polchinski flow (2.2) depends on the fields via functional derivatives of S — linearly through the operators $S^{(2)}$ and $P_\Lambda^{-1}\,\phi\,S^{(1)}$, and non-linearly through $S^{(1)}\,S^{(1)}$. Each of these terms can grow large for large fields and/or large couplings, even in the momentum regime where $\partial_t P_\Lambda$ is non-vanishing. In general, they do not cancel amongst each other. Hence, the right-hand side of (2.2) remains non-trivial in large parts of field space. This makes it more difficult, within given truncations, to identify stable flows or regions in field space where the flow remains small. On the other hand, the polynomial non-linearities in (2.2) are simpler than those in (2.1) as they do not involve an inverse of $S^{(n)}$, also leading to a smaller number of different terms in the flow for a n-point function $\partial_t S^{(n)}$ for sufficiently large n. These aspects have been considered as a major benefit of the Polchinski flow, e.g. [14]. Below, we shall see that it is precisely the non-linear dependence of (2.1) on the inverse of $\Gamma_k^{(2)}$ which implies stability of the flow, whereas the polynomial non-linearities in (2.2) are responsible for less stable solutions.

Flow equations based on a proper-time regularisation have received considerable interest recently [16 − 19]. They derive from a proper-time regularisation of the effective action valid to one-loop order [16]. Any proper-time flow at momentum scale k can be represented in the basis of

$$\partial_t\Gamma_k[\phi] = \frac{1}{2}\,\mathrm{Tr}\left(\frac{k^2}{k^2 + \Gamma_k^{(2)}/m}\right)^m, \quad (2.3)$$

or linear combinations thereof, where the parameter $m \geq 1$ characterises the momentum cutoff [18, 13]. Path integral derivations of proper-time flows have been worked out in [18, 13, 19]. In their simplest form, they make use of background fields, where plain momenta q^2 in the cutoff is replaced by $\Gamma_k^{(2)}[\phi]$ evaluated at some background field [13]. Therefore, an exact proper-time flow contains additional flow terms proportional to $\partial_t\Gamma_k^{(2)}$ on its right-hand side [19]. Implicit to this approach is that differences between fluctuation and background field are neglected. The proportionality of (2.3) to (powers of) the full propagator is responsible for an increased suppression of the flow at large fields or couplings with increasing m. In this respect, proper-time flows are similar to ERG flows and,

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consequently, allow for analogous optimisations [8, 20]. Given the established link between exact proper time flows and ERG flows, it suffices, in the remainder, to elaborate on the differences between (2.1) and (2.2).

3. Derivative expansion

Both (2.1) and (2.2) are exact flows. In consequence, the physical content of the fully integrated flows should be identical. This equivalence, in general, cannot be maintained within specific approximations, unavoidable as soon as either method is applied to a non-trivial physical problem. In order to highlight the similarities and differences between (2.1) and (2.2), we study both flows for an $\mathrm{O}(N)$ symmetric real scalar field ϕ^a, $a = 1, \ldots, N$, in d dimensions within a derivative expansion [21], which is the most commonly used expansion scheme for critical phenomena (see also [10, 22]). A priori, little is known about its convergence as there is no small expansion parameter associated to it [20]. The ERG flow for Γ_k, and the Polchinski flow for S_Λ are linked by a Legendre transform and additional momentum-dependent field rescalings. This implies that derivative expansions for the ERG and the Polchinski RG are inequivalent. To leading order in the derivative expansion, an Ansatz for the effective action Γ_k contains a standard kinetic term and the effective potential U_k,

$$\Gamma_k = \int d^d x \left(U_k(\bar\rho) + \frac{1}{2}\partial_\mu\phi^a\partial_\mu\phi_a \right), \tag{3.1}$$

where $\bar\rho = \frac{1}{2}\phi^a\phi_a$. Introducing dimensionless variables $\rho = \bar\rho k^{2-d}$, $U(\bar\rho) = u(\rho)\, k^d$, and using (2.1), the flow equation for the effective potential is

$$\partial_t u + du - (d-2)\rho u' = (N-1)\,\ell(\omega_1) + \,\ell(\omega_2) \tag{3.2}$$

with $\omega_1 = u'$ and $\omega_2 = u' + 2\rho u''$. The function $\ell(\omega)$ encodes the non-trivial flow, and reads

$$\ell(\omega) = v_d \int_0^\infty dy\, y^{d/2}\, \frac{\partial_t r(y)}{y(1+r)+\omega} \tag{3.3}$$

with $y \equiv q^2/k^2$, $r(y) = R(q^2)/q^2$, $\partial_t r(y) = -2yr'(y)$, and $v_d^{-1} = 2^{d+1}\pi^{d/2}\Gamma(d/2)$. The flow (3.2) is a second order non-linear partial differential equation. All non-trivial information regarding the renormalisation flow and the regularisation scheme are encoded in the function (3.3). The momentum integration is peaked and regularised for large momenta due to the cutoff term $\partial_t r(y)$, and for small momenta due to $r(y)$ in the numerator.

All terms on the left-hand side of (3.2) are cutoff independent, and display the intrinsic scaling of the variables that have been chosen for the parametrisation of the flow. By making use of rescaling in the fields and in the effective potential, the numerical factor $\sim v_d$ can be removed. Rescalings of the fields and the infrared scale parameter cannot remove the explicit cutoff dependence in (3.3). The R-dependence of the flow (3.2) can be characterised by appropriate moments of R [7].

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In [7 − 9, 20], ideas have been put forward to increase the stability and physical content of truncated RG flows by choosing 'optimised' regulators R. The main observation is that the infrared cutoff R, in addition to regularising the flow, also controls its convergence and stability properties. This fact entails that specific cutoffs lead to improved results already at a fixed order in a systematic expansion. An optimisation then corresponds to identifying the RG flows with best stability properties. As an example, consider the flow (2.1) in an expansion in vertex functions about vanishing field. The truncated propagator is $G_k(q^2) = (q^2 + R)^{-1}$. Its largest contribution to the flow stems from the momentum range where $G_k(q^2)$ is maximal. In units of k, its maximum $C^{-1}(R) = \max_{q^2/k^2} \left[G_k(q^2)\, k^2\right]$ depends on the cutoff function R. Taking R such that $C(R)$ is as large as possible stabilises the flow. Furthermore, higher order corrections are suppressed by C^{-1}. Consequently, maximising the "gap parameter" $C(R)$ corresponds to an optimisation,

$$C_{\max} = \max_R \; C(R)\,, \tag{3.4}$$

and cutoffs R which obey $C(R) = C_{\max}$ are refered to as solutions of the optimisation condition. This condition requires an appropriate normalisation for the cutoff, typically $R(q^2 = c\,k^2) = c\,k^2$ for some $c > 0$ [7, 8]. To leading order, optimised cutoffs are independent of the specific theory studied. Note that (3.4) is a very mild condition: it only fixes one parameter in R out of infinitely many. This implies that the subspace of optimised cutoffs is still infinite dimensional. For more details, we refer to [7 − 9].

Optimised flows have a number of interesting properties: their radius of convergence for amplitude expansions is increased [7], they factorise thermal and quantum fluctuations in the flow [8], they entail a minimum sensitivity condition [9], they improve the derivative expansion [20], they lead to a fast decoupling of heavy modes, and they lead to an improved approach to convexity for theories with spontaneous symmetry breaking. An important optimised cutoff is given by [8]

$$R_{\rm opt}(q^2) = (k^2 - q^2)\,\theta(k^2 - q^2)\,. \tag{3.5}$$

In momentum space, the cutoff (3.5) is distinguished because it solves (3.4) in the entire domain $q^2 < k^2$, and not only at the minimum of the inverse cutoff propagator. Hence, (3.5) implements the gap criterion in a global manner. In the space of all optimised cutoff propagators, the cutoff (3.5) corresponds to the convex hull of optimised inverse cutoff propagators (cf. figure 1 in [8]), reflecting the extremal property of (3.5). In more physical terms, the cutoff (3.5) leaves the propagation of large momentum modes $q^2 > k^2$ unchanged, $q^2 + R \approx q^2$. In turn, the propagation of infrared modes with $q^2 < k^2$ is cut off leading to an effective mass term $q^2 + R \approx k^2$. When expressed in terms of (3.5), the flow (3.2) becomes

$$\partial_t u + du - (d-2)\rho u' = \frac{N-1}{1+u'} + \frac{1}{1+u'+2\rho u''}\,. \tag{3.6}$$

The numerical factor $4v_d/d$ has been absorbed into the potential and the fields by an appropriate rescaling.

Now we turn to the Polchinski flow. We use an Ansatz for $S_\Lambda[\phi]$ analogous to (3.1). For comparison with (3.2), we introduce the dimensionless effective potential $u(\rho) = U_\Lambda/\Lambda^d$ and the dimensionless field variable $\rho = \frac{1}{2}\phi^a\phi_a\Lambda^{2-d}$. Then, for $N \neq 0$, the Polchinski flow reads [14, 23, 24]

$$\partial_t u - du + (d-2)\rho u' = u' + \frac{2}{N}\rho u'' - 2\rho(u')^2\,. \tag{3.7}$$

We also have performed a finite renormalisation of the fields and the potential. The crucial observation at this point is that the flow equation (3.7) is cutoff independent. The main differences and similarities between the ERG flow (3.2), (3.6) and the Polchinski flow (3.7) are summarised as follows:

(*i*) *Non-linearities* — The essential nonlinearities in the flows (3.2) and (3.7) are very different. For the Polchinski flow, they reduce to a quadratic term $\sim \rho\cdot(u')^2$. For the ERG flow, the non-linearities appear solely in a denominator, a direct consequence of the structural form of (2.1). When expanded in powers of the fields, the flow contains all and arbitrarily high powers of u', $2\rho u''$ and products thereof.

(*ii*) *Stability* — The structure of the non-linearities influence the stability properties of the flows. In the strong coupling domain or at large fields, the non-trivial part of the ERG flow (the right-hand side of (3.2)) is small, effectively suppressed by powers of the propagator. The non-linearities of the Polchinski flow (the right-hand side of (3.7)) are unbounded for large fields and couplings.

(*iii*) *Cutoff independence* — The terms on the right-hand sides of (3.2) and (3.7) originate from the non-trivial flow of the potential and contain the essential non-linearities. Their structure constitutes the main qualitative difference between the two flows. The ERG flow (3.2) depends on infinitely many moments of the regulator R. This can be seen explicitly by expanding the flow in powers of ω_1 and ω_2. On a scaling solution, the Polchinski flow (3.7) is fully scheme independent. This is in marked contrast to the ERG flow, where the regulator dependence cannot be removed by rescalings of the fields. A weak scheme dependence may persist even after an integration of the truncated flow.

4. Universality

Next, we switch to $d = 3$ dimensions and compare the Polchinski and ERG flows on a quantitative level in the vicinity of a scaling solution, the Wilson-Fisher fixed point. The numerical values for universal critical exponents from Polchinski flows, due to their scheme independence to leading order in a derivative expansion, can be seen as a benchmark test for any other approach in the same approximation. For ERG flows, this check is non-trivial since (3.2) depends on the cutoff.

Universal critical exponents ν and subleading corrections-to-scaling exponents have been deduced in the literature from the non-trivial scaling solution $\partial_t u'_* = 0$ as eigenvalues λ of small perturbations $\delta u'$ around the fixed point, $\partial_t(u'_*+\delta u') = \partial_t\,\delta u' = \lambda\,\delta u'$ ($\nu = -1/\lambda_0$, where λ_0 is the single negative eigenvalue). The flow $\partial_t\,\delta u'$ and the eigenperturbations $\delta u'$ depend on the scaling solution u'_*.

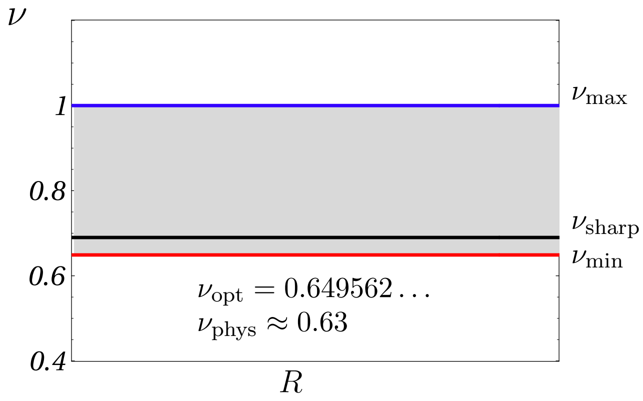

Figure 1: Critical exponent ν for various R, $N = 1$. The shaded region contains about 10^3 data points for various classes of cutoffs. We have indicated the sharp cutoff result (black line), the numerical result $\nu_{\rm max}$ at the upper boundary (blue line) and at the lower boundary $\nu_{\rm min} = \nu_{\rm opt}$ (red line). The latter coincides with the result from the optimised ERG flow (3.6) and the Polchinski flow (3.7). The physical value is also indicated (see text).

We begin with the ERG flow (3.2). The maximal ranges of attainable values for the critical exponents $\nu(R)$, for all N, has been discussed in [25]. The main result is depicted in figure 1 for $N = 1$, which contains $\sim 10^3$ data points for $\nu(R)$, based on qualitatively different classes of cutoff functions including exponential, compact and algebraic ones, combinations thereof, discontinuous cutoffs and cutoffs with sliding scales. The main result represented by figure 1 is that the range of values for $\nu(R)$ is bounded both from above and from below. The numerical value at the upper bound

$$\nu_{\rm max} = \max_R \ \nu(R)\,, \tag{4.1}$$

corresponds to the large-N limit $\nu = 1$. The upper boundary is achieved for a Callan-Symanzik type flow with mass-like regulator $R \to k^2$. In this limit, the corresponding flow ceases to be a wilsonian flow in the strict sense because the momentum integration in (2.1) is no longer cut-off in the ultraviolet limit. In the light of the optimisation, these flows have poor convergence and stability properties, and do not represent solutions to (3.4). This behaviour is equally reflected in the increasingly poor numerical convergence of solutions to (3.2) for cutoffs with ν in the vicinity of (4.1). Conversely, in the vicinity of the lower boundary

$$\nu_{\rm min} = \min_R \ \nu(R)\,, \tag{4.2}$$

the flows have good convergence and stability behaviour. Results $\nu(R)$ from generic optimised flows are typically less than 1% away from the lower boundary (4.2). Hence, the numerical value $\nu_{\rm min}$ and all regularisations leading to it, are distinguished. Most interestingly, the minimum (4.2) is achieved for the cutoff (3.5),

$$\nu_{\rm opt} \stackrel{!}{=} \nu_{\rm min}\,. \tag{4.3}$$

Therefore, the value (4.3) has maximal reliability in the present truncation and is taken as the physical prediction to this order.

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N	Polchinski	R_{opt}
0	—	0.592083^a
1	$0.64956^{c,e}$	0.649562^a
2	0.7082^d	0.708211^a
3	0.7611^d	0.761123^a
4	0.8043^d	0.804348^a

Table 1: Critical exponent ν (see text).

Figure 1 can also be interpreted in the light of the principle of minimum sensitivity (PMS) [26]. We emphasize that the latter is only applicable because $\nu(R)$ is globally bounded. Then, the PMS condition corresponds to the choice of cutoffs R_{PMS}, for which $\delta\nu(R)/\delta R$ vanishes. For one-parameter families of cutoffs $R(b)$ parametrised by b, the PMS condition often has several solutions for b_{PMS} and ν_{PMS}, e.g. [27, 9]. Requiring that $\nu(R)$ is 'globally' extremal with respect to the regularisation identifies both boundaries (4.1) and (4.3) as solutions of a global minimum sensitivity condition. Here, 'global' refers to the fact that the extrema in figure 1 are achieved within the entire space of regulators R, and not just 'locally' for some n-parameter subclasses thereof (see [25]). Hence, the PMS condition by itself, neither locally nor globally, is sufficient to provide a unique physical prediction for ν. In turn, the optimisation condition singles out a unique prediction: locally, for one-parameter families of cutoffs, it leads to values for ν in the vicinity of (4.3) [20]; globally, the value (4.3) is singled out straightaway. Hence, the optimisation which has let to the choice (3.5) is equivalent to a global extremisation of $\nu(R)$. This provides an explicit example for the more general result of [9], which states that the optimisation entails a minimum sensitivity condition, while the converse, in general, is not true.

We continue with a comparison of all critical exponents that have been published to date within both the optimised ERG flow (3.6) and the Polchinski flow (3.7). These are: the critical exponent ν (table 1), the smallest correction-to-scaling exponent ω (table 2) and the asymmetric corrections-to-scaling exponent ω_5 (table 3). Based on the optimised ERG flow (3.6), the exponents ν and ω have been given for all N in [25] with up to six significant figures (a). The exponent ω_5 has been computed in [28] for $N = 1$ (b). Cutoff independence of the Polchinski flow implies unique results for universal eigenvalues. The critical exponents ν and ω have been computed with up to four significant digits in [14] for $N = 1$ (c), and in [23] for $N = 1, \ldots, 4$ (d). Results up to five digits for the exponents ν, ω and ω_5 have recently been stated in [29] for $N = 1$ (e). Except for $N = \infty$, there are no published results based on the Polchinski flow for $N > 4$. However, it has recently been indicated [30] that the results from Polchinski flow and optimised ERG flow also agree for $N > 4$. In the large N limit, the spread of $\nu(R)$ with R is absent, and the results for critical exponents becomes unique, $\nu(R) = 1$, and $\omega_n(R) = 2n - 1$, $n = 1, 2, \ldots$ in agreement with the corresponding limit of the Polchinski flow [31].

Hence, it is most remarkable that all universal critical exponents computed either from the Polchinski flow or from the optimised ERG flow, agree to all significant figures, for the leading and subleading critical exponents, and for different universality classes! This high

N	Polchinski	$R_{\rm opt}$
0	—	0.65788^a
1	$0.65574^{c,e}$	0.655746^a
2	0.6712^d	0.671221^a
3	0.6998^d	0.699837^a
4	0.7338^d	0.733753^a

Table 2: Subleading correction-to-scaling exponent ω.

degree of coincidence leads to the important conjecture that the universal content of the partial differential equations (3.6) and (3.7) is equivalent.[1]

5. Stability

Next, we analyse the locality and stability structure of flows and show that the non-universal properties of RG flows in the vicinity of a scaling solution are vastly different. Consider the fixed point solution itself, which is non-universal and not measurable in any experiment. Using (3.7), the nontrivial scaling solution with $u'_\star \neq$ const. and $\partial_t u'_\star = 0$ of the Polchinski RG obeys the differential equation

$$2u'_\star - (d-2)\rho u''_\star = 2(u'_\star)^2 - \frac{2}{N}\rho u'''_\star - \left(1 + \frac{2}{N} - 4\rho u'_\star\right) u''_\star \,. \tag{5.1}$$

For large fields, the scaling potential behaves as

$$u_\star(\rho) \propto \rho + \text{subleading} \,. \tag{5.2}$$

An analytical solution for $N = \infty$ has been given in [32]. On the level of the RG flow, this behaviour stems from a cancellation between the canonical scaling of the potential and its non-linear renormalisation. From (5.2), we conclude that the non-trivial quantum contributions to the Polchinski flow diverges like

$$\partial_t u - du + (d-2)\rho u' \propto \rho + \text{subleading} \tag{5.3}$$

for large fields ρ close to a scaling solution. Within the optimised ERG, the non-trivial scaling solution, using (3.6), obeys

$$2u'_\star - (d-2)\rho u''_\star = -(N-1)\frac{u''_\star}{(1+u'_\star)^2} - \frac{3u''_\star + 2\rho u'''_\star}{(1 + u'_\star + 2\rho u''_\star)^2} \,. \tag{5.4}$$

Analytical solutions for the limit $N = \infty$ have been given in [33]. For $N \neq \infty$, and in the vicinity of $\rho = 0$, the scaling solution can be obtained analytically as a Taylor expansion in the field [25]. In the limit of large fields $\rho \gg 1$, we find

$$u_\star(\rho) \propto \rho^{1+\alpha} + \text{subleading} \tag{5.5}$$

[1] After publication, an explicit map between (3.6) and (3.7) has been worked out by T.R. Morris in hep-th/0503161.

N	Polchinski	$R_{\rm opt}$
1	1.8867^{e}	1.8867^{b}

Table 3: Asymmetric correction-to-scaling exponent ω_5.

for arbitrary regulator, where $\alpha = 2/(d-2)$ is positive for $d > 2$. Here, the large-field behaviour is solely due to the canonical scaling dimension of the fields. From (5.4), and for large fields $\rho \gg 1$, we conclude that the non-linear part of the ERG flow for the potential behaves as

$$\partial_t u + du - (d-2)\rho u' \propto \rho^{-\alpha} + \text{subleading}\,. \tag{5.6}$$

Hence, the right-hand side of (5.6) is suppressed for all $d > 2$.

More generally, the result (5.6) holds for generic ERG flows where the right-hand side of (3.2) decays $\propto 1/\omega$ for large ω. The power-law behaviour is altered for cutoffs which effectively introduce non-localities due to their momentum structure, e.g. the mass-like cutoff (no large momentum decay), the sharp cutoff, or cutoffs with an algebraic large-momentum decay like the quartic cutoff $R \sim k^4/q^2$, e.g. [34]. They lead, respectively, to (5.6) with a large-field behaviour $\propto \rho$, $\propto \ln\rho$ and, in three dimensions, $\propto \rho^{-3/2}$. The different power law exponents $\alpha(R)$ as a function of the cutoff are displayed in figure 2 for three dimensions. The minimum

$$\alpha_{\rm min} = \min_R\ \alpha(R) \tag{5.7}$$

is achieved for Callan-Symanzik type flows and the Polchinski flow, $\alpha_{\rm min} = -1 < 0$. A negative α also indicates that an additional renormalisation of the flow is necessary due to an insufficiency in the integrating-out of momentum modes. This is well-known for Callan-Symanzik type flows [6]. We stress, however, that the set of flows with negative α is of measure zero; generic ERG flows have positive α. The sharp cutoff marks the boundary between ERG flows with insignificant ($\alpha > 0$) and significant ($\alpha < 0$) contributions for large fields, and hence the boundary between flows which are essentially local, respectively non-local, in the fields. The maximum

$$\alpha_{\rm max} = \max_R\ \alpha(R) \tag{5.8}$$

is achieved for generic ERG flows including optimised ones, $\alpha_{\rm max} = 2/(d-2) > 0$. Note that the few 'non-local' ERG flows with α in the range $[\alpha_{\rm min}, \alpha_{\rm sharp}]$ lead to critical exponents ν in figure 1 in the range $[\nu_{\rm max}, \nu_{\rm sharp}]$, whereas all 'local' ERG flows with α within $[\alpha_{\rm sharp}, \alpha_{\rm max}]$ — the overwhelming majority of all ERG flows — lead to values within the narrow window $[\nu_{\rm sharp}, \nu_{\rm min}]$ [25]. We conclude that flows with underlying non-localities have the tendency to deviate strongly from the physical theory and display a strong cutoff dependence, while local flows display only a weak cutoff dependence, thereby remaining close to the physical theory. In the Ising universality class, the exponent ν from non-local (local) flows deviates between 10-50% (3-10%) from the physical value. This quantifies the link between the locality structure of the flow, its stability, and its vicinity to the physical theory.

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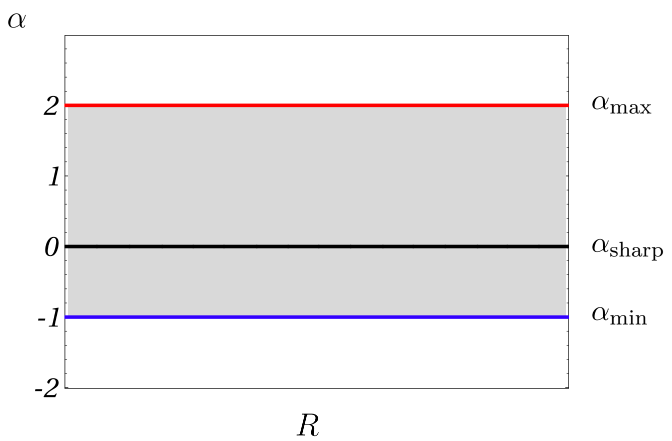

Figure 2: Large-field behaviour of wilsonian flows close to a critical point, $d = 3$. The shaded region indicates the range of values for $\alpha(R)$ for various R. The sharp cutoff (black line) marks the boundary between flows with significant ($\alpha < 0$) and insignificant ($\alpha > 0$) contributions at large fields. Callan-Symanzik type flows and the Polchinski flow have $\alpha = \alpha_{\rm min}$ (blue line), generic ERG flows have $\alpha = \alpha_{\rm max}$ (red line). Flows with an increased locality structure lead to improved results (see text).

Summarising, unlike the universal parts the non-universal scaling solutions derived from ERG or Polchinski flows are vastly different. This result also extends to the non-universal eigenperturbations at criticality. The non-trivial quantum corrections to the flow at large fields are strongly suppressed for generic ERG fows (5.6), while they remain large in the Polchinski case (5.3). This is a direct consequence of the structural differences in the basic flows (2.1) and (2.2). Despite of having the same universal content, in the light of figure 2 the optimised flow (3.6) and the Polchinski flow (3.7) have maximally distinct locality structures. Note that good locality properties of flows are at the root for stable numerical integrations, and the quantitative smallness of quantum corrections in large domains of field space improves the convergence of RG flows. Based on the result that ERG flows with increasing non-localities show an increasingly strong cutoff dependence, we expect that Polchinski flows display a similar behaviour as soon as the leading-order degeneracy with respect to the cutoff is lifted by higher order operators in extended truncation.

6. Extensions

A global analysis of critical exponents in the full space of cutoffs has only been performed to leading order in the derivative expansion, which is the most important order as higher order effects should be suppressed proportional to the anomalous dimension η of the order of a few percent. Still, it is useful to briefly review the results achieved so far beyond leading order in the light of the preceeding discussion.

Within Polchinski flows, the leading order scheme independence is lost as soon as higher order derivative operators are taken into account [14]. To order $\mathcal{O}(\partial^n)$, the flow depends explicitely on n scheme-dependent parameters which cannot be removed by further rescalings. To order $\mathcal{O}(\partial^2)$, a part of the cutoff dependence has been probed for different projections on the anomalous dimension, i.e. [14, 24, 29, 30]. All published results for ν and η to order $\mathcal{O}(\partial^2)$ have in common that the spurious dependence on remaining unphysical cutoff parameters is monotonous, without displaying local extrema. The range of numerical results includes the physical values. Unfortunately, none of the truncations admits a minimum sensitivity condition, and further conditions have to be invoked to remove the scheme dependence. From a structural point of view, the comparatively strong cutoff dependence beyond leading order is not unexpected and fully in line with the stability considerations detailed in the preceeding section. It remains to be seen whether the next order in the expansion has a stabilising effect on the series [30].

Within the ERG, parts of the cutoff space have been probed quantitatively to order $\mathcal{O}(\partial^2)$ [35 − 37] and to order $\mathcal{O}(\partial^4)$ [38]. Two observations have to be made in the present context: first of all, the critical exponents ν and η remain bounded, similar to figure 1, in the parameter range considered. Furthermore, they attain local extrema as functions of the cutoff indicating the existence of a global boundary equivalent to those displayed in figure 1. The boundedness of $\nu(R)$ within a given order of the derivative expansion is an important ingredient in the convergence of the series. Secondly, the set of stable flows, as identified through the optimisation, remains stable to higher orders: typically, flows with regulators R such that $\nu(R)$ to leading order is in the vicinity of $\nu_{\rm opt}$ remain in the vicinity of the local extrema of $\nu(R)$ even beyond leading order. This confirms the validity of the underlying optimisation. For sufficiently stable flows, higher order corrections remain quantitatively small, thus increasing the convergence of the derivative expansion [20].

Finally, we point out that the qualitative differences beyond leading order are also reflected in the explicit cutoff dependence of either flow. ERG flows depend only on 'global' properties of the regulator R through specific momentum integrals $a_n(R)$ of the form

$$a_n(R) \sim \int_0^\infty dy \frac{-y^{d/2+1-n}\, r'(y)}{[1+r(y)]^n}$$

and similar [7]. This follows from expanding the flow (3.2) in powers of u' and $u'+2\rho u''$. The coefficients $a_n(R)$ receive their main contributions for momenta $q^2 \approx k^2$. Therefore, small changes in the momentum behaviour of the cutoff $R \to R + \delta R$ induce small changes in all coefficients $a_n(R)$. Furthermore, many different cutoff functions R can lead to equivalent sets of coefficients a_n. Hence, the precise small- or large-momentum structure of R is at best of subleading relevance to the flow. In turn, Polchinski flows in a derivative expansion depend both on 'global' and on 'local' characteristics of the regulator $K(q^2/\Lambda^2)$, and in particular on its derivatives at vanishing momenta. For example, the Polchinski flow for the wave function renormalisation depends on the ratio $B(K) = K''(0)/K'(0)^2$, and the anomalous dimension at criticality η is even proportional to B [14] (see also [24, 29, 30]). Small modifications in the cutoff $K \to K + \delta K$ at small momenta can induce large changes in B including its sign, and, therefore, induce comparatively large alterations in the flow and the physical observables. These structural differences can be seen as a further indication for the increased stability of ERG flows as opposed to Polchinski flows.

7. Discussion and conclusions

We compared several functional renormalisation group equations based on wilsonian cutoffs. The main structural differences between Polchinski flows and ERG flows are due to the non-linearities of their right-hand sides. In the literature, it has sometimes been argued that the simple non-linearities of the Polchinski flow (2.2) as opposed to those of ERG flows (2.1) are a benefit to the formalism. Here, we arrived at the opposite conclusion: the non-linearities in ERG flows involve the inverse of $\Gamma_k^{(2)}$, and, therefore, guarantee that the flow remains small in large regions of field and momentum space. This structure implies that ERG flows are amiable to systematic expansions (e.g. in vertex functions) and allow for a straightforward optimisation, since truncational variations in the flow are suppressed in large parts of field space. On the other side, the non-linearities of the Polchinski flow appear to be algebraically simpler. The price to pay is that the flow remains non-trivial in a larger domain of field space, including the region of large fields. These differences in the locality and stability behaviour favour the flows (2.1) in particular for numerical implementations.

The structural differences have been made explicit within a derivative expansion. To leading order, critical exponents from the Polchinski flow are scheme independent and, therefore, serve as a benchmark test for functional RG flows in corresponding approximations. We have established the remarkable result that the optimised ERG flow (3.6) and the Polchinski flow (3.7) have identical universal eigenvalues, for all $\mathrm{O}(N)$ symmetric scalar theories, for the leading and subleading critical exponents, and for the asymmetric corrections-to-scaling exponent! This equivalence is non-trivial in that the corresponding flows (3.6) and (3.7) differ substantially, both in their structure and in their non-universal scaling solutions. We conjecture that this result extends to all universal observables to leading order in the derivative expansion.

This equivalence, however, does not persist in an obvious manner beyond the leading order, where universal observables from Polchinski flows depend strongly on remaining unphysical parameters. This is in marked contrast to the results from ERG flows which remain bounded, similar to the leading order. The comparatively large cutoff sensitivity and the nontriviality of the Polchinski flow for large fields — a consequence of the non-linearities in (2.2) — require a better conceptual understanding before definite conclusions can be drawn concerning its convergence properties.

For ERG flows, on the other hand, a coherent picture has emerged. In given truncations, an appropriate optimisation leads to an increased stability of the flow. The cutoff dependence of physical observables is, thereby, largely reduced to a small range in the vicinity of the physical theory. This pattern is established quantitatively within a derivative expansion, both to leading order and beyond. The comparatively weak cutoff sensitivity of optimised flows and the triviality of the flow for large fields — a consequence of the non-linearities in (2.1) — are at the root for reliable applications of the formalism to more complex theories including QCD and gravity.

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